

GL(2)-STRUCTURES IN DIMENSION FOUR, H-FLATNESS AND INTEGRABILITY

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ABSTRACT. We show that torsion-free four-dimensional GL(2)-structures are flat up to a coframe transformation with a mapping taking values in a certain subgroup $H \subset \mathrm{SL}(4, \mathbb{R})$ which is isomorphic to a semidirect product of the three-dimensional continuous Heisenberg group $H_3(\mathbb{R})$ and the Abelian group \mathbb{R} . In addition, we show that the relevant PDE system is integrable in the sense that it admits a dispersionless Lax-pair.

1. INTRODUCTION

A GL(2)-structure on a smooth 4-manifold M is given by a smoothly varying family of twisted cubic curves, one in each projectivised tangent space of M . Equivalently, a GL(2)-structure is the same as G -structure $\pi: B \rightarrow M$ on M , where G is the image subgroup of the faithful irreducible 4-dimensional representation of $\mathrm{GL}(2, \mathbb{R})$ on the space of homogeneous polynomials of degree three with real coefficients in two real variables. In particular, a GL(2)-structure is called *torsion-free* if its associated G -structure is torsion-free. Torsion-free GL(2)-structures are of particular interest as they provide examples of torsion-free connections with exotic holonomy group $\mathrm{GL}(2, \mathbb{R})$. However, the local existence of torsion-free GL(2)-structures is highly non-trivial, even when applying the Cartan–Kähler machinery which is particularly well-suited for the construction of torsion-free connections with special holonomy. Adapting methods of Hitchin [9], Bryant [2] gave an elegant twistorial construction of real-analytic torsion-free GL(2)-structures in dimensions four, thus providing the first example of an irreducibly-acting holonomy group of a (non-metric) torsion-free connection missing from Berger’s list [1] of such connections.

A natural source for GL(2)-structures are differential operators. Recall that the principal symbol $\sigma(D)$ of a k -th order linear differential operator $D: C^\infty(M, \mathbb{R}^n) \rightarrow C^\infty(M, \mathbb{R}^m)$ assigns to each point $p \in M$ a homogeneous polynomial of degree k on T_p^*M with values in $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$. Therefore, in each projectivised cotangent space $\mathbb{P}(T_p^*M)$ of M we obtain the so-called *characteristic variety* Ξ_p of D , consisting of those $[\xi] \in \mathbb{P}(T_p^*M)$ for which the linear mapping $\sigma_\xi(D): \mathbb{R}^n \rightarrow \mathbb{R}^m$ fails to be injective. In particular, given a (possibly non-linear) differential operator D and a smooth

\mathbb{R}^n -valued function u defined on some open subset $U \subset M$ and which satisfies $D(u) = 0$, we may ask that the linearisation $L_u(D)$ of D around u has characteristic varieties all of which are the dual variety of the twisted cubic curve. Consequently, one obtains a $GL(2)$ -structure on the domain of definition of each solution u of the PDE $D(u) = 0$ for an appropriate class of differential operators. Various examples of such operators have recently been given by Ferapontov–Kruglikov [6]. In particular, they show that locally all torsion-free $GL(2)$ -structures arise in this fashion for some *second order* operator D which furthermore has the property that the PDE $D(u) = 0$ admits a dispersionless Lax pair. We also refer the reader to [7] for an application of similar ideas to the case of three-dimensional Einstein–Weyl structures.

Here we show that if a 4-manifold M carries a torsion-free $GL(2)$ -structure $\pi: B \rightarrow M$, then for every point $p \in M$ there exists a p -neighbourhood U_p , local coordinates $x: U_p \rightarrow \mathbb{R}^4$ and a mapping $h: U_p \rightarrow H$ into a certain 4-dimensional subgroup $H \subset SL(4, \mathbb{R})$, so that the coframing $\eta = h dx$ is a local section of $\pi: B \rightarrow M$. Moreover, the mapping h satisfies a *first order* quasi-linear PDE system which admits a dispersionless Lax-pair. Note that our result shows that 4-dimensional torsion-free $GL(2)$ -structures are *H-flat*, that is, flat up to a coframe transformation with a mapping taking values in H .

Along the way (see Theorem 2.1), we derive a first order quasi-linear PDE describing general *H-flat* torsion-free *G*-structures which may be of independent interest.

Acknowledgments. The authors would like to thank Maciej Dunajski and Evgeny Ferapontov for helpful conversations and correspondence.

2. *G*-STRUCTURES AND *H*-FLATNESS

In this section we collect some elementary facts about *G*-structures, introduce the notion of *H-flatness* and derive the first order PDE system describing *H-flat* torsion-free *G*-structures. Throughout the article all manifolds and maps are assumed to be smooth, that is C^∞ .

2.1. The coframe bundle and *G*-structures. Let M be an n -manifold and V a real n -dimensional vector space. A coframe at $p \in M$ is a linear isomorphism $f: T_p M \rightarrow V$. The set $F_p M$ of coframes at $p \in M$ is the fibre of the principal right $GL(V)$ coframe bundle $v: FM \rightarrow M$ where the right action $R_a: FM \rightarrow FM$ is defined by the rule $R_a(f) = a^{-1} \circ f$ for all $a \in GL(V)$ and $f \in FM$. Of course, we may identify $V \simeq \mathbb{R}^n$, but it is often advantageous to allow V to be an abstract vector space in which case we say FM is *modelled* on V . The coframe bundle carries a tautological V -valued 1-form defined by $\omega_f = f \circ v'$ so that we have the equivariance property $R_a^* \omega = a^{-1} \omega$. A local v -section $\eta: U \rightarrow FM$ is called a *coframing* on $U \subset M$ and a choice of a basis of V identifies η with n linearly independent 1-forms on U .

Let $G \subset \mathrm{GL}(V)$ be a closed subgroup. A G -structure on M is a reduction $\pi: B \rightarrow M$ of the coframe bundle with structure group G , equivalently a section of the fibre bundle $FM/G \rightarrow M$. Note that V is equipped with a canonical coframing η_0 defined by the exterior derivative of the identity map $\eta_0 = d\mathrm{Id}_V$. Consequently, the coframe bundle of V may naturally be identified with $V \times \mathrm{GL}(V)$ and hence the space of G -structures on V is in one-to-one correspondence with the maps $V \rightarrow \mathrm{GL}(V)/G$. In particular, a map $h: V \rightarrow \mathrm{GL}(V)$ defines a G -structure on V by composing h with the quotient projection $\mathrm{GL}(V) \rightarrow \mathrm{GL}(V)/G$.

2.2. H -flatness. A G -structure $\pi: B \rightarrow M$ is called *flat* if in a neighbourhood U_p of every point $p \in M$ there exist local coordinates $x: U_p \rightarrow V$ so that $dx: U_p \rightarrow FM$ takes values in B . Suppose $H \subset \mathrm{GL}(V)$ is a closed subgroup. We say a G -structure is *H -flat* if in a neighbourhood U_p of every point $p \in M$ there exist local coordinates $x: U_p \rightarrow V$ and a mapping $h: U_p \rightarrow H$ so that $h dx: U_p \rightarrow FM$ takes values in B . Clearly, every G -structure is $\mathrm{GL}(V)$ -flat and a G -structure is flat in the usual sense if and only if it is $\{e\}$ -flat where $\{e\}$ denotes the trivial subgroup of $\mathrm{GL}(V)$.

Example 2.1. Every $\mathrm{O}(2)$ -structure is \mathbb{R}^+ -flat where \mathbb{R}^+ denotes the group of uniform scaling transformations of \mathbb{R}^2 with positive scale factor. This is the existence of local isothermal coordinates for Riemannian metrics in two-dimensions. Likewise, conformally flat Riemannian metrics yield examples of $\mathrm{O}(n)$ -structures that are \mathbb{R}^+ -flat.

2.3. A PDE for H -flat torsion-free G -structures. Recall that a G -structure $\pi: B \rightarrow M$ is called *torsion-free* if there exists a principal G -connection θ on B so that Cartan's first structure equation

$$(1) \quad d\omega = -\theta \wedge \omega$$

holds.

Remark 2.2. We remark that a weaker notion of torsion-freeness is also in use, see for instance [3, 10]. Namely, a G -structure $\pi: B \rightarrow M$ is called torsion-free if there exists a pseudo-connection on B with vanishing torsion, that is, a 1-form θ on B with values in the Lie algebra of G so that (1) holds.

We may ask when a G -structure on V induced by a mapping $h: V \rightarrow H \subset \mathrm{GL}(V)$ is torsion-free. To this end let $A \subset V^* \otimes V$ be a linear subspace. Denote by

$$\delta: V^* \otimes V^* \otimes V \rightarrow \Lambda^2(V^*) \otimes V$$

the natural skew-symmetrisation map. Recall that the Spencer cohomology group $H^{0,2}(A)$ of A is the quotient

$$H^{0,2}(A) = (\Lambda^2(V^*) \otimes V) / \delta(V^* \otimes A).$$

Let

$$\Pi_A: \Lambda^2(V^*) \otimes V \rightarrow H^{0,2}(A)$$

denote the quotient projection and let μ_H denote the Maurer-Cartan form of H . Note that $\psi_h = h^* \mu_H$ is a 1-form on V with values in the Lie algebra \mathfrak{h} of H , that is, a smooth map

$$\psi_h: V \rightarrow V^* \otimes \mathfrak{h} \subset V^* \otimes \mathfrak{gl}(V) \simeq V^* \otimes V^* \otimes V.$$

We define $\tau_h = \delta \psi_h$, so that τ_h is a 2-form on V with values in V . Moreover, let $\text{Ad}(h): \mathfrak{gl}(V) \rightarrow \mathfrak{gl}(V)$ denote the adjoint action of $h \in H$. We now have:

Theorem 2.1. *Let $h: V \rightarrow H$ be a smooth map. Then the G -structure defined by h is torsion-free if and only*

$$(2) \quad \Pi_{\text{Ad}(h^{-1})\mathfrak{g}} \tau_h = 0.$$

Remark 2.3. In the case where $H = G$ the H -structure defined by h is the same as the torsion-free H -structure defined by the map $h \equiv \text{Id}_V: V \rightarrow \text{GL}(V)$, hence (2) must be trivially satisfied. This is indeed the case, for any map $h: V \rightarrow H$ we obtain

$$\Pi_{\text{Ad}(h^{-1})\mathfrak{h}} \tau_h = \Pi_{\mathfrak{h}} \tau_h = \Pi_{\mathfrak{h}} \delta \psi_h = 0,$$

since the adjoint action of H preserves \mathfrak{h} .

Proof of Proposition 2.1. For the proof we fix an identification $V \simeq \mathbb{R}^n$. Let $x = (x^i)$ denote the standard coordinates on \mathbb{R}^n . Furthermore let $h: \mathbb{R}^n \rightarrow H \subset \text{GL}(n, \mathbb{R})$ be given and let $\pi: B_h \rightarrow \mathbb{R}^n$ denote the G -structure defined by h , that is,

$$B_h = \{(x, a) \in \mathbb{R}^n \times \text{GL}(n, \mathbb{R}) : a = h^{-1}(x)g, g \in G\}.$$

Note that we have an G -bundle isomorphism

$$\psi: \mathbb{R}^n \times G \rightarrow B_h, \quad (x, g) \mapsto (x, h^{-1}(x)g).$$

The tautological 1-form ω_0 on $F\mathbb{R}^n \simeq \mathbb{R}^n \times \text{GL}(n, \mathbb{R})$ satisfies $(\omega_0)_{(x,a)} = a^{-1}dx$ for all $(x, a) \in \mathbb{R}^n \times \text{GL}(n, \mathbb{R})$. Continuing to write ω_0 for the pullback to B_h of ω_0 , we obtain

$$\omega_{(x,g)} := (\psi^* \omega_0)_{(x,g)} = g^{-1}h(x)dx.$$

Let α be any 1-form on \mathbb{R}^n with values in \mathfrak{g} , the Lie-algebra of G . We obtain a principal G -connection $\theta = (\theta_j^i)$ on $\mathbb{R}^n \times G$ by defining

$$\theta = g^{-1}\alpha g + g^{-1}dg$$

where $g: \mathbb{R}^n \times G \rightarrow G \subset \text{GL}(n, \mathbb{R})$ denotes the projection onto the later factor. Conversely, every principal G -connection on the trivial G -bundle $\mathbb{R}^n \times G$ arises in this fashion. The G -structure B_h is torsion-free if and only if there exists a principal G -connection θ such that

$$d\omega + \theta \wedge \omega = 0$$

which is equivalent to

$$0 = d(g^{-1}hdx) + (g^{-1}\alpha g + g^{-1}dg) \wedge g^{-1}hdx$$

or

$$0 = (dg^{-1} + g^{-1}dg g^{-1}) \wedge h dx + g^{-1} (dh \wedge dx + \alpha \wedge h dx)$$

Using $0 = d(g^{-1}g)$ we see that the G -structure defined by h is torsion-free if and only if there exists a 1-form α on V with values in \mathfrak{g} such that

$$0 = dh \wedge dx + \alpha \wedge h dx.$$

This is equivalent to

$$(h^{-1}dh + h^{-1}\alpha h) \wedge dx = 0$$

or

$$(3) \quad (\psi_h + \text{Ad}(h^{-1})\alpha) \wedge dx = 0$$

where $\psi_h = h^{-1}dh$ denotes the h -pullback of the Maurer-Cartan form of H and $\text{Ad}(h)v = hvh^{-1}$ the adjoint action of $h \in H$ on $v \in \mathfrak{gl}(n, \mathbb{R})$. Now (3) is equivalent to

$$\delta \psi_h + \delta \text{Ad}(h^{-1})\alpha = 0$$

Since α takes values in \mathfrak{g} this implies that $\tau_h = \delta \psi_h$ lies in the δ -image of $V^* \otimes \text{Ad}(h^{-1})\mathfrak{g}$ which implies

$$\Pi_{\text{Ad}(h^{-1})\mathfrak{g}} \tau_h = 0.$$

Conversely, suppose τ_h lies in the δ -image of $V^* \otimes \text{Ad}(h^{-1})\mathfrak{g}$. Then there exists a 1-form β on V with values in $h^{-1}\mathfrak{g}h$ so that

$$\tau_h = \delta \psi_h = \delta \beta.$$

Hence the \mathfrak{g} -valued 1-form α on V defined by $\alpha = -h\beta h^{-1}$ satisfies

$$\tau_h + \delta h^{-1}\alpha h = \delta \psi_h + \delta \text{Ad}(h^{-1})\alpha = 0,$$

thus proving the claim. \square

3. GL(2)-STRUCTURES

Let x, y denote the standard coordinates on \mathbb{R}^2 and let $\mathbb{R}[x, y]$ denote the polynomial ring with real coefficients generated by x and y . We let $\text{GL}(2, \mathbb{R})$ act from the left on $\mathbb{R}[x, y]$ via the usual linear action on x, y . We denote by \mathcal{V}_d the subspace consisting of homogeneous polynomials in degree $d \geq 0$ and by $G_d \subset \text{GL}(\mathcal{V}_d)$ the image subgroup of the $\text{GL}(2, \mathbb{R})$ action on \mathcal{V}_3 . The vector space \mathcal{V}_3 carries a two-dimensional cone $\tilde{\mathcal{C}}$ of distinguished polynomials consisting of the perfect cubes, i.e., those that are of the form $(ax + by)^3$ for $ax + by \in \mathcal{V}_1$. The reader may easily check that G_3 is characterised as the subgroup of $\text{GL}(\mathcal{V}_3)$ that preserves $\tilde{\mathcal{C}}$. The projectivisation of $\tilde{\mathcal{C}}$ gives an algebraic curve \mathcal{C} of degree 3 in $\mathbb{P}(\mathcal{V}_3)$ which is linearly equivalent to the *twisted cubic curve*, i.e., the curve in \mathbb{RP}^3 defined by the zero locus of the three homogeneous polynomials

$$P_0 = XZ - Y^2, \quad P_1 = YW - Z^2, \quad P_2 = XW - YZ,$$

where $[X:Y:Z:W]$ are the standard homogeneous coordinates on \mathbb{RP}^3 .

Let M be a 4-manifold and let $v: FM \rightarrow M$ denote its coframe bundle modelled on \mathcal{V}_3 . A $\mathrm{GL}(2)$ -structure on M is a reduction $\pi: B \rightarrow M$ of FM with structure group $G_3 \simeq \mathrm{GL}(2, \mathbb{R})$. By definition, a $\mathrm{GL}(2)$ -structure identifies each tangent space of M with \mathcal{V}_3 up to the action by $\mathrm{GL}(2, \mathbb{R})$. Consequently, each projectivised tangent space $\mathbb{P}(T_p M)$ of M carries an algebraic curve \mathcal{C}_p which is linearly equivalent to the twisted cubic curve. Conversely, if $\mathcal{C} \subset \mathbb{P}(TM)$ is a smooth subbundle having the property that each fibre \mathcal{C}_p is linearly equivalent to the twisted cubic curve, then one obtains a unique reduction of the coframe bundle of M whose structure group is G_3 .

For what follows it will be convenient to identify $\mathcal{V}_3 \simeq \mathbb{R}^4$ by the isomorphism $\mathcal{V}_3 \rightarrow \mathbb{R}^4$ defined on the basis of monomials as

$$x^{(3-i)}y^i \mapsto e_{i+1}$$

where $i = 0, 1, 2, 3$ and e_i denotes the standard basis of \mathbb{R}^4 . Note that, under the identification $T_p M = \mathcal{V}_3$ the cone $\tilde{\mathcal{C}}$ of a $\mathrm{GL}(2)$ -structure at p can be written as

$$\tilde{\mathcal{C}}_p = \{s^3 e_1 + 3s^2 t e_2 + 3s t^2 e_3 + t^3 e_4 \mid s, t \in \mathbb{R}\}.$$

We now have:

Theorem 3.1. *All torsion-free $\mathrm{GL}(2)$ -structures in dimension four are H -flat, where $H \subset \mathrm{SL}(4, \mathbb{R})$ is the subgroup consisting of matrices of the form*

$$(4) \quad \begin{pmatrix} 1 & A & B & D \\ 0 & 1 & A & C \\ 0 & 0 & 1 & A \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where A, B, C, D are arbitrary real numbers.

Remark 3.1. We note that the group H is isomorphic to a semidirect product of the continuous three-dimensional Heisenberg group $H_3(\mathbb{R})$ and the Abelian group \mathbb{R} , that is, $H \simeq H_3(\mathbb{R}) \rtimes \mathbb{R}$. Indeed, $H_3(\mathbb{R})$ has a faithful (necessarily reducible) four-dimensional representation defined by the Lie group homomorphism $\varphi: H_3(\mathbb{R}) \rightarrow \mathrm{GL}(4, \mathbb{R})$

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & a & \frac{1}{2}a^2 + b & \frac{1}{6}a^3 + ab - c \\ 0 & 1 & a & \frac{1}{2}a^2 \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that φ embeds $H_3(\mathbb{R})$ as a normal subgroup of the group H and we think of \mathbb{R} as the subgroup of H defined by setting $A = B = D = 0$ in (4).

Remark 3.2. In fact, the notion of a $\mathrm{GL}(2)$ -structure makes sense in all dimensions $d \geq 3$. However, torsion-free $\mathrm{GL}(2)$ -structures in dimensions exceeding four are $\{e\}$ -flat [2], that is, flat in the usual sense. We refer the reader to [8, 16] for a comprehensive study of five-dimensional $\mathrm{GL}(2)$ -structures (with torsion).

Remark 3.3. Phrased differently, Theorem 3.1 states that locally every torsion-free GL(2)-structure in dimension four is obtained from a solution to the quasi-linear first order PDE system (2) where h takes values in the aforementioned group H .

Proof of Theorem 3.1. We shall prove that for a given torsion-free GL(2)-structure one can always choose local coordinates such that the cone $\tilde{\mathcal{C}}$ has the following form

$$\tilde{\mathcal{C}} = \{ s^3 V_0 + 3s^2 t V_1 + 3st^2 V_2 + t^3 V_3 \mid s, t \in \mathbb{R} \}$$

where the framing (V_0, V_1, V_2, V_3) is

$$(5) \quad \begin{aligned} V_0 &= \partial_0, & V_1 &= \partial_1 + \alpha \partial_0, & V_2 &= \partial_2 + \alpha \partial_1 + \beta \partial_0, \\ V_3 &= \partial_3 + \alpha \partial_2 + \gamma \partial_1 + \delta \partial_0, \end{aligned}$$

for some functions α, β, γ and δ . Then, the dual coframing is of the form $h dx$, where h takes values in H . Indeed, we have

$$h = \begin{pmatrix} 1 & -\alpha & -\beta + \alpha^2 & -\delta + \alpha(\gamma + \beta) - \alpha^3 \\ 0 & 1 & -\alpha & -\gamma + \alpha^2 \\ 0 & 0 & 1 & -\alpha \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In order to derive the desired form of $\tilde{\mathcal{C}}$ we explore a correspondence between the torsion-free GL(2)-structures and classes of contact equivalent fourth order ODEs (compare the proof of [4, Theorem 1] and a similar correspondence in dimension 3). Indeed, it is proved in [2] that any torsion-free GL(2)-structure is defined by a fourth order ODEs of the form

$$(6) \quad x^{(4)} = F(y, x, x', x'', x''')$$

satisfying the Bryant-Wünschmann condition which is a system of two non-linear equations for a unknown function $F = F(y, x_0, x_1, x_2, x_3)$ (see also [5, 12, 15]). Here, (y, x_0, x_1, x_2, x_3) are the standard coordinates on the space $J^3(\mathbb{R}, \mathbb{R})$ of 3-jets of functions $\mathbb{R} \rightarrow \mathbb{R}$. The Bryant-Wünschman condition is invariant with respect to the group of contact transformations of variables (y, x_0, x_1, x_2, x_3) . The GL(2)-structure corresponding to equation (6) is defined on the solution space of (6), i.e. on the quotient space $J^3(\mathbb{R}, \mathbb{R})/X_F$, where $X_F = \partial_y + x_1 \partial_0 + x_2 \partial_1 + x_3 \partial_2 + F \partial_3$ is the total derivative. In order to define the structure we first consider the following field of cones on $J^3(\mathbb{R}, \mathbb{R})$ as in [11]

$$\hat{\mathcal{C}} = \{ s^3 \hat{V}_0 + 3s^2 t \hat{V}_1 + 3st^2 \hat{V}_2 + t^3 \hat{V}_3 \mid s, t \in \mathbb{R} \} \mod X_F$$

where

$$\begin{aligned}
\hat{V}_0 &= \frac{3}{4}\partial_3, & \hat{V}_1 &= \frac{1}{2}\partial_2 + \frac{3}{8}\partial_3 F \partial_3 \\
\hat{V}_2 &= \frac{1}{2}\partial_1 + \frac{1}{4}\partial_3 F \partial_2 + \left(\frac{7}{20}\partial_2 F - \frac{3}{20}X_F(\partial_3 F) + \frac{9}{40}(\partial_3 F)^2 \right) \partial_3 \\
\hat{V}_3 &= \partial_0 + \frac{1}{4}\partial_3 F \partial_1 + \left(\partial_2 F + \frac{7}{10}K \right) \partial_2 \\
&\quad + \left(\partial_1 F - \frac{3}{10}X_F(K) - X_F(\partial_2 F) + \frac{21}{40}K \partial_3 F \right. \\
&\quad \left. - \frac{27}{16}X_F(\partial_3 F)\partial_3 F - \frac{3}{4}\partial_2 F \partial_3 F + \frac{3}{4}X_F^2(\partial_3 F) + \frac{27}{64}(\partial_3 F)^3 \right) \partial_3,
\end{aligned}$$

and $K = -\partial_2 F + \frac{3}{2}X(\partial_3 F) - \frac{3}{8}(\partial_3 F)^2$. To define the cone one looks for (f, g) such that

$$(7) \quad \text{ad}_{fX_F}^4(g\partial_3) = 0 \mod X_F, \partial_3, \partial_2$$

where $\text{ad}_{X_F}^i$ stands for the iterated Lie bracket with vector field X_F . Then $\hat{\mathcal{C}}_p$ is defined as the set of all $(\text{ad}_{fX_F}^3(g\partial_3))(p)$ where (f, g) solve (7). The explicit formula for $\hat{\mathcal{C}}$ can be found using [11, Proposition 4.1] and [11, Corollary 5.3]. The cone $\hat{\mathcal{C}}$ is invariant with respect to the flow of X_F if and only if (6) satisfies the Bryant-Wünschmann condition. In this case (7) takes the form $\text{ad}_{fX_F}^4(g\partial_3) = 0 \mod X_F$ (c.f. [12]). Then $\hat{\mathcal{C}}$ can be projected to the quotient space $J^3(\mathbb{R}, \mathbb{R})/X_F$ and defines a $\text{GL}(2)$ -structure there via the field of cones $\tilde{\mathcal{C}} = q_*\hat{\mathcal{C}}$, where $q: J^3(\mathbb{R}, \mathbb{R}) \rightarrow J^3(\mathbb{R}, \mathbb{R})/X_F$ is the quotient map. Note that $J^3(\mathbb{R}, \mathbb{R})/X_F$ can be identified with the hypersurface $\{y = 0\} \subset J^3(\mathbb{R}, \mathbb{R})$. Denoting

$$\begin{aligned}
\alpha &= \partial_3 F|_{y=0}, \\
\beta &= \left(\frac{7}{20}\partial_2 F - \frac{3}{20}X(\partial_3 F) + \frac{9}{40}(\partial_3 F)^2 \right) \Big|_{y=0}, \\
\gamma &= \left(\partial_2 F + \frac{7}{10}K \right) \Big|_{y=0}, \\
\delta &= \left(\partial_1 F - \frac{3}{10}X(K) - X(\partial_2 F) + \frac{21}{40}K \partial_3 F - \frac{27}{16}X(\partial_3 F)\partial_3 F \right. \\
&\quad \left. - \frac{3}{4}\partial_2 F \partial_3 F + \frac{3}{4}X^2(\partial_3 F) + \frac{27}{64}(\partial_3 F)^3 \right) \Big|_{y=0}
\end{aligned}$$

we get that

$$\tilde{\mathcal{C}} = \{ s^3 V_0 + 3s^2 t V_1 + 3st^2 V_2 + t^3 V_3 \mid s, t \in \mathbb{R} \}$$

where

$$\begin{aligned} V_0 &= \frac{3}{4}\partial_3, & V_1 &= \frac{1}{2}\partial_2 + \frac{3}{8}\alpha\partial_3, & V_2 &= \frac{1}{2}\partial_1 + \frac{1}{4}\alpha\partial_2 + \beta\partial_3, \\ V_3 &= \partial_0 + \frac{1}{4}\alpha\partial_1 + \gamma\partial_2 + \delta\partial_3. \end{aligned}$$

The following linear change of coordinates

$$(x_0, x_1, x_2, x_3) \mapsto \left(x_3, 2x_2, 2x_1, \frac{4}{3}x_0\right)$$

transforms (V_0, V_1, V_2, V_3) to

$$\begin{aligned} V_0 &= \partial_0, & V_1 &= \partial_1 + \frac{1}{2}\alpha\partial_0, & V_2 &= \partial_2 + \frac{1}{2}\alpha\partial_1 + \frac{4}{3}\beta\partial_0, \\ V_3 &= \partial_3 + \frac{1}{2}\alpha\partial_2 + 2\gamma\partial_1 + \frac{4}{3}\delta\partial_0, \end{aligned}$$

which is equivalent to (5) up to constants. \square

Remark 3.4. It is proved in [6] that locally any torsion-free GL(2)-structure admits a coframing of the form $h \, dx$ with

$h =$

$$\begin{pmatrix} a_1a_2a_3 & a_0a_2a_3 & a_0a_1a_3 & a_0a_1a_2 \\ \frac{1}{3}(a_1a_2b_3 + a_1b_2a_3 + b_1a_2a_3) & \frac{1}{3}(a_0a_2b_3 + a_0b_2a_3 + b_0a_2a_3) & \frac{1}{3}(a_0a_1b_3 + a_0b_1a_2 + b_0a_1a_3) & \frac{1}{3}(a_0a_1b_2 + a_0b_1a_2 + b_0a_1a_2) \\ \frac{1}{3}(a_1b_2b_3 + b_1a_2b_3 + b_1b_2a_3) & \frac{1}{3}(a_0b_2b_3 + b_0a_2b_3 + b_0b_2a_3) & \frac{1}{3}(a_0b_1b_3 + b_0a_1b_3 + b_0b_1a_3) & \frac{1}{3}(a_0b_1b_2 + b_0a_1b_2 + b_0b_1b_2) \end{pmatrix},$$

where $a_i = \left(\frac{\partial u}{\partial x_i}\right)^{-1}$ and $b_i = \left(\frac{\partial v}{\partial x_i}\right)^{-1}$ for some real-valued functions u and v on $\mathcal{V}_3 \simeq \mathbb{R}^4$. It is an interesting problem to find the smallest possible dimension of the group H , such that all torsion-free GL(2)-structures are H -flat.

4. INTEGRABILITY

In this section we derive the system (2) explicitly in terms of the functions A, B, C and D of Theorem 3.1. Moreover, we prove that it possesses a dispersionless Lax pair understood as a pair of commuting vector fields depending on a spectral parameter. Systems of this type, e.g. the dispersionless Kadomtsev-Petviashvili equation, often appear as dispersionless limits of integrable PDEs. Other examples include the Plebański heavenly equation or the Manakov-Santini system describing 3-dimensional Einstein-Weyl geometry. We refer to [13, 14] for general methods of integration of such systems.

Theorem 4.1. *Let $H \subset \mathrm{SL}(4, \mathbb{R})$ be the subgroup of matrices (4). An H -flat GL(2)-structure defined by a coframing $h \, dx$, where h takes values in H ,*

satisfies (2), i.e. is torsion-free, if and only if

$$\begin{aligned}
 & V_2(D) - V_3(B) - AV_2(B) - CV_2(A) + AV_3(A) + A^2V_2(A) = 0 \\
 & 2V_1(D) - V_2(C) - 2AV_1(B) - V_3(A) + \\
 & \quad + AV_2(A) + 2A^2V_1(A) - 2CV_1(A) = 0 \\
 (8) \quad & V_0(D) - 2V_1(C) + 3V_1(B) - AV_0(B) - 2V_2(A) \\
 & \quad - AV_1(A) - CV_0(A) + A^2V_0(A) = 0 \\
 & V_0(C) - 2V_0(B) + V_1(A) + AV_0(A) = 0,
 \end{aligned}$$

where (V_0, V_1, V_2, V_3) is the framing dual to $h \, dx$ explicitly given by

$$\begin{aligned}
 V_0 &= \partial_0, & V_1 &= \partial_1 - A\partial_0, & V_2 &= \partial_2 - A\partial_1 - (B - A^2)\partial_0, \\
 V_3 &= \partial_3 - A\partial_2 - (C - A^2)\partial_1 - (D - (C + B)A + A^3)\partial_0.
 \end{aligned}$$

The system (8) can be put in the following Lax form

$$[L_0, L_1] = 0,$$

with

$$\begin{aligned}
 L_0 &= \partial_3 + (-C + 2A\lambda - 3\lambda^2)\partial_1 \\
 &\quad + (-D + AC - 2A^2\lambda + 4A\lambda^2 - 2\lambda^3)\partial_0 + N(\lambda)\partial_\lambda, \\
 L_1 &= \partial_2 + (-A + 2\lambda)\partial_1 + (-B + A^2 - 2A\lambda + \lambda^2)\partial_0 + M(\lambda)\partial_\lambda
 \end{aligned}$$

and

$$\begin{aligned}
 N(\lambda) &= \left(\frac{1}{2}A^2A_1 - ABA_0 + AA_2 - AB_1 - \frac{1}{2}DA_0 - \frac{1}{2}C_2 \right. \\
 &\quad \left. + \frac{1}{2}AC_1 + \frac{1}{2}BC_0 - \frac{1}{2}CA_1 + \frac{1}{2}ACA_0 + \frac{1}{2}A_3 \right) \\
 &\quad + (3B_1 - C_1 - AA_1 - AC_0 + 2BA_0 - 2A_2)\lambda \\
 &\quad + (C_0 - A_1)\lambda^2 \\
 M(\lambda) &= \left(\frac{1}{2}AA_1 + \frac{1}{2}AC_0 - BA_0 + A_2 - B_1 \right) \\
 &\quad + \left(\frac{1}{2}A_1 - \frac{1}{2}C_0 \right) \lambda,
 \end{aligned}$$

where A_i, B_i, C_i and D_i denote $\partial_i A, \partial_i B, \partial_i C$ and $\partial_i D$, respectively, and λ is an auxiliary spectral coordinate.

Remark 4.1. The spectral parameter λ can be treated as an affine parameter on the fibres of \mathcal{C} . The theorem states that $\mathcal{D} = \text{span}\{L_0, L_1\}$ is an integrable rank-2 distribution on \mathcal{C} . There is a 3-parameter family of integral manifolds of \mathcal{D} . Projections of these submanifolds to M give a 3-parameter family of 2-dimensional submanifolds of M tangent to the field of cones $\tilde{\mathcal{C}}$.

Remark 4.2. A Cartan–Kähler analysis reveals that the first order system (8) – or equivalently (2) – is involutive and has solutions depending on four functions of three variables, confirming the count of Bryant [2]. Moreover, easy computations show that the characteristic variety of system (8) linearised along any solution (A, B, C, D) is the discriminant locus, i.e. the tangential variety of $\tilde{\mathcal{C}}$.

Proof of Theorem 4.1. The system (8) can be directly obtained by expanding (2) explicitly in terms of the functions A, B, C, D . Here we use a different method and apply [11, Corollary 7.4] to the framing $(V_0, 3V_1, 3V_2, V_3)$. Namely, denoting $\lambda = \frac{s}{t}$ we get that the curve \mathcal{C} in $\mathbb{P}(TM)$ is the image of $\lambda \mapsto \mathbb{R}V(\lambda) \in \mathbb{P}(TM)$, where $V(\lambda) = \lambda^3 V_0 + 3\lambda^2 V_1 + 3\lambda V_2 + V_3$ and the vector fields V_0, V_1, V_2 and V_3 are given by (5) with

$$\alpha = -A, \quad \beta = -B + A^2, \quad \gamma = -C + A^2, \quad \delta = -D + (C + B)A - A^3.$$

According to [11, Corollary 7.2], a GL(2)-structure is torsion-free if and only if

$$(9) \quad \left[V(\lambda), \frac{d}{d\lambda} V(\lambda) \right] \in \text{span} \left\{ V(\lambda), \frac{d}{d\lambda} V(\lambda), \frac{d^2}{d\lambda^2} V(\lambda) \right\},$$

for any $\lambda \in \mathbb{R}$. This, due to [11, Corollary 7.4] applied to the framing $(V_0, 3V_1, 3V_2, V_3)$, is expressed as eight linear equations for structural functions c_{ij}^k defined by $[V_i, V_j] = \sum_k c_{ij}^k V_k$. However, in the present case the vector fields V_i are special and four equations are void. Indeed, the nontrivial equations are as follows:

$$c_{23}^0 = 0, \quad c_{23}^1 - 2c_{13}^0 = 0$$

and

$$c_{23}^2 - 2c_{13}^1 + c_{03}^0 + 3c_{12}^0 = 0, \quad c_{23}^3 - 2c_{13}^2 + c_{03}^1 + 3c_{12}^1 - 2c_{02}^0 = 0$$

(the equations differ from equations in [11] because of the factor 3 next to V_1 and V_2 in the present paper). Substituting the structural functions, which can be easily computed, we get the system (8).

Now, we consider

$$L_0 = V(\lambda) - \left(\lambda - \frac{1}{3}A \right) \frac{d}{d\lambda} V(\lambda) \quad \text{mod } \partial_\lambda$$

and

$$L_1 = \frac{1}{3} \frac{d}{d\lambda} V(\lambda) \quad \text{mod } \partial_\lambda.$$

Due to (9), the commutator $[L_0, L_1]$ lies in the span of $\{L_0, L_1, \frac{d^2}{d\lambda^2} V(\lambda)\}$ mod ∂_λ . Moreover, since

$$L_0 = \partial_3 \quad \text{mod } \partial_1, \partial_0, \partial_\lambda$$

and

$$L_1 = \partial_2 \quad \text{mod } \partial_1, \partial_0, \partial_\lambda$$

we get $[L_0, L_1] = \varphi \frac{d^2}{d\lambda^2} V(\lambda) \bmod \partial_\lambda$ for some φ . One checks by direct computations that $N(\lambda)$ and $M(\lambda)$ are chosen such that $\varphi = 0$ and the coefficient of $[L_0, L_1]$ next to ∂_λ vanishes as well. \square

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